Observation estimate for kinetic transport equation by diffusion approximation

Claude Bardos, Kim Dang Phung

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Abstract

We study the unique continuation property for the neutron transport equation and for a simplified model of the Fokker-Planck equation in a bounded domain with absorbing boundary condition. An observation estimate is derived. It depends on the smallness of the mean free path and the frequency of the velocity average of the initial data. The proof relies on the well known diffusion approximation under convenience scaling and on basic properties of this diffusion. Eventually we propose a direct proof for the observation at one time of parabolic equations. It is based on the analysis of the heat kernel.

1 Introduction

This article is devoted to the question of unique continuation for linear kinetic transport equation with a scattering operator in the diffusive limit. Let Ω be a bounded open subset of $\mathbb{R}^{\mathrm{d}}$, d>1, with boundary $\partial\Omega$ of class C^2. Consider in $\left\{(x,v)\in\Omega\times\mathbb{S}^{\mathrm{d}-1}\right\}\times\mathbb{R}_t^+$ the transport equation in the v direction with a scattering operator S and absorbing boundary condition

$$\begin{cases} \partial_t f + \dfrac{1}{\epsilon} v\cdot\nabla f + \dfrac{a}{\epsilon^2} S\left(f\right) = 0 & \text{in } \Omega\times\mathbb{S}^{\mathrm{d}-1}\times(0,+\infty)\ , \\ f=0 & \text{on } \left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)_-\times(0,+\infty)\ , \\ f\left(\cdot,\cdot,0\right) = f_0 \in L^2(\Omega\times\mathbb{S}^{\mathrm{d}-1})\ , & \end{cases} \tag{1.1}$$

where $\epsilon\in(0,1]$ is a small parameter and $a\in L^\infty\left(\Omega\right)$ is a scattering opacity satisfying $0<c_{min}\leq a\left(x\right)\leq c_{max}<\infty$. Here, $\nabla=\nabla_x$ and $\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)_-=\left\{(x,v)\in\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}; v\cdot\vec{n}_x<0\right\}$ where $\vec{n}_x$ is the unit outward normal field at $x\in\partial\Omega$.

Two standard examples of scattering operators S are the following:

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- The neutron scattering operator:

$$S = f - \langle f\rangle \text{ where } \langle f\rangle (x,t) = \frac{1}{|\mathbb{S}^{\mathrm{d}-1}|}\int_{\mathbb{S}^{\mathrm{d}-1}} f(x,v,t)\, dv \ .$$

- The Fokker-Planck scattering operator:

$$S = -\frac{1}{\mathrm{d}-1}\Delta_{\mathbb{S}^{\mathrm{d}-1}} f \text{ where } \Delta_{\mathbb{S}^{\mathrm{d}-1}} \text{ is the Laplace-Beltrami operator on } \mathbb{S}^{\mathrm{d}-1}.$$

Let ω be a nonempty open subset of Ω. Suppose we observe the solution f at time $T > 0$ and on ω, i.e. $f(x,v,T)_{|(x,v)\in\omega\times\mathbb{S}^{\mathrm{d}-1}}$ is known. A classical inverse problem consists to recover at least one solution, and in particular its initial data, which fits the observation on $\omega \times \mathbb{S}^{\mathrm{d}-1} \times \{T\}$. Our problem of unique continuation is: With how many initial data, the corresponding solution achieves the given observation $f(x,v,T)_{|(x,v)\in\omega\times\mathbb{S}^{\mathrm{d}-1}}$. Here ϵ is a small parameter and it is natural to focus on the limit solution. This is the diffusion approximation saying that the solution f converges to a solution of a parabolic equation when ϵ tends to 0 (see [B],[DL],[LK],[BR],[BGPS],[BSS],[BBGS]). In this framework, two remarks are in order:

- For our scattering operator, there holds

$$\|f - \langle f\rangle\|_{L^2(\Omega\times\mathbb{S}^{\mathrm{d}-1}\times\mathbb{R}_t^+)} \leq \epsilon\frac{1}{\sqrt{2c_{min}}}\|f_0\|_{L^2(\Omega\times\mathbb{S}^{\mathrm{d}-1})} \ .$$

  For the operator of neutron transport, one uses a standard energy method by multiplying both sides of the first line of (1.1) by f and integrating over $\Omega \times \mathbb{S}^{\mathrm{d}-1} \times (0,T)$. For the Fokker-Planck scattering operator, one combines the standard energy method as above and Poincaré inequality

$$\begin{aligned}&\|f - \langle f\rangle\|_{L^2(\Omega\times\mathbb{S}^{\mathrm{d}-1}\times\mathbb{R}_t^+)} \\ &\leq \tfrac{1}{\sqrt{\mathrm{d}-1}}\|\nabla_{\mathbb{S}^{\mathrm{d}-1}} f\|_{L^2(\Omega\times\mathbb{S}^{\mathrm{d}-1}\times\mathbb{R}_t^+)} \leq \epsilon\tfrac{1}{\sqrt{2c_{min}}}\|f_0\|_{L^2(\Omega\times\mathbb{S}^{\mathrm{d}-1})} \ .\end{aligned}$$

- In the sense of distributions in Ω, for any $t \geq 0$, the average of f solves the following parabolic equation

$$\begin{aligned}&\partial_t\langle f\rangle - \tfrac{1}{\mathrm{d}}\nabla\cdot\left(\tfrac{1}{a}\nabla\langle f\rangle\right) \\ &= \nabla\cdot\left(\tfrac{1}{a}\langle(v\otimes v)\nabla(f-\langle f\rangle)\rangle\right) + \epsilon\nabla\cdot\left(\tfrac{1}{a}\langle v\partial_t f\rangle\right) \ .\end{aligned} \tag{1.2}$$

  Indeed, multiply by $\dfrac{\epsilon}{a}v$ the equation $\partial_t f + \dfrac{1}{\epsilon}v\cdot\nabla f + \dfrac{a}{\epsilon^2}Sf = 0$ and take the average over $\mathbb{S}^{\mathrm{d}-1}$, using $\partial_t\langle f\rangle + \dfrac{1}{\epsilon}\langle v\cdot\nabla f\rangle = 0$, $\langle v\cdot\nabla Sf\rangle = \langle v\cdot\nabla f\rangle$ and $\langle v(v\cdot\nabla\langle f\rangle)\rangle = \frac{1}{\mathrm{d}}\nabla\langle f\rangle$, one obtains for any $t \geq 0$ and any $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \partial_t\langle f\rangle\varphi dx + \frac{1}{\mathrm{d}}\int_\Omega \frac{1}{a}\nabla\langle f\rangle\cdot\nabla\varphi dx + \int_\Omega \frac{1}{a}\langle v(v\cdot\nabla(f-\langle f\rangle) + \epsilon\partial_t f)\rangle\cdot\nabla\varphi dx = 0 \ .$$

  Moreover, we prove that the boundary condition on $\langle f\rangle$ is small in some adequate norm with respect to ϵ. In the sequel, any estimates will be explicit with respect to ϵ.

Backward uniqueness for parabolic equation has a long history (see [DJP],[V]). Lions and Malgrange [LM] used the method of Carleman estimates. Later, Bardos and Tartar [BT] gave some improvements by using the log convexity method of Agmon and Nirenberg. More recently, motivated by control theory and inverse problems (see [I],[P]), Carleman estimates became an important tool to achieve an observability inequality (see [FI],[FZ],[FG],[LRL],[LeRR],[LRR]). In [PW], the desired observability inequality is deduced from the observation estimate at one point in time which is obtained by studying the frequency function in the spirit of log convexity method. In particular, one can quantify the following unique continuation property (see [EFV],[PWa]): If $u(x,t) = e^{t\Delta} u_0(x)$ with $u_0 \in L^2(\Omega)$ and $u(\cdot,T) = 0$ on ω, then $u_0 \equiv 0$.

Our main result below involves the regularity of the nonzero initial data f_0 measured in term of two quantities. Let $p > 2$,

$$\mathbb{M}_p := \frac{\|f_0\|_{L^{2p}\left(\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)}}{\|\langle f_0\rangle\|_{L^2(\Omega)}} \text{ and } \mathbb{F} := \frac{\|\langle f_0\rangle\|^2_{L^2(\Omega)}}{\|\langle f_0\rangle\|^2_{H^{-1}(\Omega)}} .$$

Observe in particular that $\mathbb{F}$ is the most natural evaluation of the frequency of the velocity average of the initial data.

Theorem 1.1 *Suppose that $a \in C^2\left(\overline{\Omega}\right)$ and $f_0 \in L^{2p}\left(\Omega \times \mathbb{S}^{\mathrm{d}-1}\right)$ with $M_p + F < +\infty$ for some $p > 2$. Then the unique solution f of (1.1) satisfies for any $T > 0$*

$$\left(1 - \epsilon^{\frac{1}{2p}}(1 + T^{\frac{p-1}{2p}} C_p)\mathbb{M}_p e^{\sigma(f_0,T)}\right) \|\langle f_0\rangle\|_{L^2(\Omega)} \leq e^{\sigma(f_0,T)} \|\langle f\rangle(\cdot,T)\|_{L^2(\omega)}$$

with $C_p = \left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}} \left(\frac{1}{p}\right)^{\frac{1}{2p}}$ and $\sigma(f_0,T) = c\left(1 + \frac{1}{T} + T\mathbb{F}\right)$ where c only depends on $(\Omega, \omega, \mathrm{d}, a)$.

By a direct application of our main result, we have:

Corollary 1.2 *Let $a \in C^2\left(\overline{\Omega}\right)$ and $f_0 \in L^{2p}\left(\Omega \times \mathbb{S}^{\mathrm{d}-1}\right)$ with $M_p + F < +\infty$ for some $p > 2$. Suppose that $f_0 \geq 0$. Then there is $\epsilon_0 \in (0,1)$ depending on $(\mathbb{M}_p, \mathbb{F}, \Omega, \omega, \mathrm{d}, p, T, a)$ such that if $f(\cdot,\cdot,T) = 0$ on $\omega \times \mathbb{S}^{\mathrm{d}-1}$ for some $\epsilon \leq \epsilon_0$, then $f_0 \equiv 0$.*

This paper is organized as follows: The proof of the main result is given in the next section. It requires two important results: an approximation diffusion convergence of the average of f; an observation estimate at one point in time for the diffusion equation with homogeneous Dirichlet boundary condition. In Section 3, we prove the approximation theorem stated in Section 2. In Section 4, a direct proof of the observation inequality at one point in time for parabolic equations is proposed. Finally, in an appendix, we prove a backward estimate for the diffusion equation and a trace estimate for the kinetic transport equation.

2 Proof of main Theorem 1.1

The main task in the proof of Theorem 1.1 consists on the two following propositions. Below we denote by $u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$ any solution of the diffusion equation

$$\partial_t u - \frac{1}{\mathrm{d}} \nabla \cdot \left(\frac{1}{a} \nabla u \right) = 0 \tag{2.1}$$

with $a \in C^2\left(\overline{\Omega}\right)$ and $0 < c_{min} \leq a(x) \leq c_{max} < \infty$.

Proposition 2.1 *There are $C > 0$ and $\mu \in (0,1)$ such that any solution $u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$ of (2.1) satisfies*

$$\int_\Omega |u(x,T)|^2 \, dx \leq \left(C e^{\frac{C}{T}} \int_\omega |u(x,T)|^2 \, dx \right)^{1-\mu} \left(\int_\Omega |u(x,0)|^2 \, dx \right)^{\mu} .$$

Here C and μ only depend on $(a, \Omega, \omega, \mathrm{d})$.

As an immediate application, combining with the following backward estimate for diffusion equation

$$\|u(\cdot,0)\|_{L^2(\Omega)} \leq c e^{cT \frac{\|u(\cdot,0)\|^2_{L^2(\Omega)}}{\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}} \|u(\cdot,T)\|_{L^2(\Omega)} \ , \tag{2.2}$$

we have:

Corollary 2.2 *For any nonzero $u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$ solution of (2.1), one has*

$$\|u(\cdot,0)\|_{L^2(\Omega)} \leq e^{C\left(1 + \frac{1}{T} + T \frac{\|u(\cdot,0)\|^2_{L^2(\Omega)}}{\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}\right)} \|u(\cdot,T)\|_{L^2(\omega)}$$

where C only depends on $(a, \Omega, \omega, \mathrm{d})$.

Proposition 2.3 *Assume $f_0 \in L^{2p}(\Omega \times \mathbb{S}^{\mathrm{d}-1})$ for some $p > 2$ and consider $u \in C\left(0,T; H_0^1(\Omega)\right)$ solution of (2.1) with initial data $u(\cdot,0) = \langle f_0 \rangle$, then for any $T > 0$ and any $\chi \in C_0^\infty(\omega)$, the solution f of (1.1) satisfies*

$$\left\| \chi \left(\langle f \rangle_{|t=T} - u(\cdot,T) \right) \right\|_{H^{-1}(\Omega)} \leq \epsilon^{\frac{1}{2p}} \left(1 + T^{\frac{p-1}{2p}} C_p \right) C \|f_0\|_{L^{2p}(\Omega \times \mathbb{S}^{\mathrm{d}-1})}$$

where $C_p = \left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}} \left(\frac{1}{p}\right)^{\frac{1}{2p}}$ and $C > 0$ only depends on $(\Omega, \mathrm{d}, a, \chi)$.

The proof of Proposition 2.1 and Proposition 2.3 is given in section 4 and section 3 respectively.

In one hand, since $u(\cdot,0)=\langle f_0\rangle$, we have by Corollary 2.2

$$\|\langle f_0\rangle\|_{L^2(\Omega)}\leq e^{C\left(1+\frac{1}{T}+T\frac{\|\langle f_0\rangle\|^2_{L^2(\Omega)}}{\|\langle f_0\rangle\|^2_{H^{-1}(\Omega)}}\right)}\|\chi u(\cdot,T)\|_{L^2(\Omega)}\ .$$

On the other hand, by regularizing effect, we have

$$\begin{aligned}\|\chi u(\cdot,T)\|_{L^2(\Omega)} &\leq \|\chi u(\cdot,T)\|^{1/2}_{H^{-1}(\Omega)}\|\chi u(\cdot,T)\|^{1/2}_{H^1_0(\Omega)}\\ &\leq C\|\chi u(\cdot,T)\|^{1/2}_{H^{-1}(\Omega)}\left(1+\tfrac{1}{T^{1/4}}\right)\|\langle f_0\rangle\|^{1/2}_{L^2(\Omega)}\ .\end{aligned}$$

Therefore, the two above facts yield

$$\begin{aligned}\|\langle f_0\rangle\|_{L^2(\Omega)} &\leq e^{C\left(1+\frac{1}{T}+T\frac{\|\langle f_0\rangle\|^2_{L^2(\Omega)}}{\|\langle f_0\rangle\|^2_{H^{-1}(\Omega)}}\right)}\left(\left\|\chi\left(u(\cdot,T)-\langle f\rangle_{|t=T}\right)\right\|_{H^{-1}(\Omega)}+\left\|\chi\langle f\rangle_{|t=T}\right\|_{H^{-1}(\Omega)}\right)\\ &\leq e^{C\left(1+\frac{1}{T}+T\frac{\|\langle f_0\rangle\|^2_{L^2(\Omega)}}{\|\langle f_0\rangle\|^2_{H^{-1}(\Omega)}}\right)}\left(\epsilon^{\frac{1}{2p}}\left(1+T^{\frac{p-1}{2p}}C_p\right)\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{\mathrm{d}-1})}+\left\|\chi\langle f\rangle_{|t=T}\right\|_{H^{-1}(\Omega)}\right)\end{aligned}$$

where in the last line we used Proposition 2.3. This completes the proof.

3 Estimates for diffusion approximation

Below we give precise error estimates for the diffusion approximation.

Theorem 3.1 *Let $a\in C^1\left(\overline{\Omega}\right)$ such that $0<c_{min}\leq a(x)\leq c_{max}<\infty$. Assume $f_0\in L^{2p}(\Omega\times\mathbb{S}^{\mathrm{d}-1})$ for some $p>2$ and consider $u\in C\left(0,T;H^1_0(\Omega)\right)$ solution of (2.1) with initial data $u(\cdot,0)=\langle f_0\rangle$, then for any $T>0$, the solution f of (1.1) satisfies*

$$\left\|\langle f\rangle_{|t=T}-u(\cdot,T)\right\|_{H^{-1}(\Omega)}+\|\langle f\rangle-u\|_{L^2(\Omega\times(0,T))}\leq\epsilon^{\frac{1}{2p}}\left(1+T^{\frac{p-1}{2p}}C_p\right)C\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{\mathrm{d}-1})}$$

where $C_p=\left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}}\left(\frac{1}{p}\right)^{\frac{1}{2p}}$ and $C>0$ only depends on (Ω,d) and $(c_{min},c_{max},\|\nabla a\|_\infty)$.

In the literature, there are at least two ways to get diffusion approximation estimates:

- Use a Hilbert expansion: The solution f of the transport problem can be formally written as $f=f_0+\epsilon f_1+\epsilon^2 f_2+...$ and we substitute this expansion into the governing equations in order to prove existence of $f_0,f_1,f_2,...$. Next we set

$F = f - (f_0 + \epsilon f_1)$ and check that it solves a transport problem for which energy method can be used. This way requires well-prepared initial data that is $f_0 = \langle f_0 \rangle$ to avoid initial layers.

- Use moment method: The zeroth and first moments of f are respectively $\langle f \rangle$ and $\langle vf \rangle$. First, we check that $f - \langle f \rangle$ is small in some adequate norm with respect to ϵ. Next by computing the zeroth and first moments of the equation solved by f (as it was done in the introduction), we derive that $\langle f \rangle$ solves a parabolic problem for which energy method can be used. This way and a new ϵ uniform estimate on the trace (see Proposition 3.2 below) give Theorem 3.1. Notice that since only the average of f, is involved, the proof requires no analysis of the initial layer near $t = 0$.

Proposition 3.2 *If $f_0 \in L^{2p}(\Omega \times \mathbb{S}^{\mathrm{d}-1})$ for some $p > 2$, then the solution f of (1.1) satisfies*

$$\|f\|_{L^2\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\times(0,T)\right)} \leq CT^{\frac{p-1}{2p}}\epsilon^{\frac{1}{2p}}C_p\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{d-1})}$$

where $C_p = \left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}}\left(\frac{1}{p}\right)^{\frac{1}{2p}}$ *and $C > 0$ only depends on* (Ω, d).

Proposition 3.2 is proved in Appendix. The proof of Theorem 3.1 starts as follows. Let $w_\epsilon = \langle f \rangle - u$ where u solves

$$\begin{cases} \partial_t u - \frac{1}{\mathrm{d}}\nabla \cdot \left(\frac{1}{a}\nabla u\right) = 0 & \text{in } \Omega \times (0, +\infty) \ , \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \ , \\ u(\cdot, 0) = \langle f_0 \rangle \in L^2(\Omega) \ . & \end{cases}$$

By (1.2) and a density argument, w_ϵ solves for any $t \geq 0$ and any $\varphi \in H_0^1(\Omega)$

$$\begin{aligned} &\int_\Omega \partial_t w_\epsilon \varphi dx + \frac{1}{\mathrm{d}}\int_\Omega \nabla w_\epsilon \cdot \frac{1}{a}\nabla\varphi dx \\ &= -\int_\Omega \langle v\left(v \cdot \nabla\left(f - \langle f \rangle\right)\right)\rangle \cdot \frac{1}{a}\nabla\varphi dx - \epsilon\int_\Omega \langle v\partial_t f\rangle \cdot \frac{1}{a}\nabla\varphi dx \end{aligned} \tag{3.1}$$

with boundary condition $w_\epsilon = \langle f \rangle$ on $\partial\Omega \times \mathbb{R}_t^+$ and initial data $w_\epsilon(\cdot, 0) = 0$. We choose

$$\varphi = \left(-\frac{1}{\mathrm{d}}\nabla \cdot \left(\frac{1}{a}\nabla\right)\right)^{-1} w_\epsilon \ .$$

By integrations by parts, the identity (3.1) becomes:

$$\begin{aligned} &\frac{1}{2\mathrm{d}}\frac{d}{dt}\int_\Omega \frac{1}{a}|\nabla\varphi|^2 dx + \left\|\frac{1}{\mathrm{d}}\nabla \cdot \left(\frac{1}{a}\nabla\right)\varphi\right\|_{L^2(\Omega)}^2 \\ &= -\int_{\partial\Omega} \langle f \rangle \frac{1}{a}\partial_n\varphi dx \\ &\quad -\int_\Omega \langle v\left(v \cdot \nabla\left(f - \langle f \rangle\right)\right)\rangle \cdot \frac{1}{a}\nabla\varphi dx \\ &\quad -\epsilon\int_\Omega \langle v\partial_t f\rangle \cdot \frac{1}{a}\nabla\varphi dx \ . \end{aligned} \tag{3.2}$$

First, the contribution of the boundary data is estimate: One has, by a classical trace theorem

$$-\int_{\partial\Omega}\langle f\rangle\frac{1}{a}\partial_n\varphi dx\le C_1\|f\|_{L^2\left(\partial\Omega\times\mathbb{S}^{d-1}\right)}\left\|\nabla\cdot\left(\frac{1}{a}\nabla\right)\varphi\right\|_{L^2(\Omega)}$$

where the constant C_1 depends on $\|\nabla a\|_\infty$.

Secondly, the contribution of the term

$$\int_\Omega\langle v\left(v\cdot\nabla\left(f-\langle f\rangle\right)\right)\rangle\cdot\frac{1}{a}\nabla\varphi dx$$

is estimated: By integration by parts and using $\nabla\varphi=\partial_n\varphi\vec{n}_x$ on $\partial\Omega$, one has

$$\begin{aligned}\int_\Omega\langle v\left(v\cdot\nabla\left(f-\langle f\rangle\right)\right)\rangle\cdot\frac{1}{a}\nabla\varphi dx &=-\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}}(f-\langle f\rangle)v\cdot\nabla\left(v\cdot\frac{1}{a}\nabla\varphi\right)dxdv\\ &+\frac{1}{|\mathbb{S}^{d-1}|}\int_{\partial\Omega\times\mathbb{S}^{d-1}}\left(v\cdot\vec{n}_x\right)^2\left(f-\langle f\rangle\right)\frac{1}{a}\partial_n\varphi dxdv\end{aligned}$$

which implies

$$\begin{aligned}\int_\Omega\langle v\left(v\cdot\nabla\left(f-\langle f\rangle\right)\right)\rangle\cdot\frac{1}{a}\nabla\varphi dx &\le C_1\|f-\langle f\rangle\|_{L^2\left(\Omega\times\mathbb{S}^{d-1}\right)}\left\|\nabla\cdot\left(\tfrac{1}{a}\nabla\right)\varphi\right\|_{L^2(\Omega)}\\ &+C_1\|f\|_{L^2\left(\partial\Omega\times\mathbb{S}^{d-1}\right)}\left\|\nabla\cdot\left(\tfrac{1}{a}\nabla\right)\varphi\right\|_{L^2(\Omega)}\end{aligned}$$

with some constant $C_1>0$ depending on $\|\nabla a\|_\infty$.

Thirdly, the contribution of the term $\epsilon\displaystyle\int_\Omega\langle v\partial_t f\rangle\cdot\frac{1}{a}\nabla\varphi dx$ is estimated: From the identities

$$\begin{aligned}&\epsilon\int_\Omega\langle v\partial_t f\rangle\cdot\frac{1}{a}\nabla\varphi dx\\ &=\frac{1}{|\mathbb{S}^{d-1}|}\epsilon\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}}fv\cdot\frac{1}{a}\nabla\varphi dxdv-\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}}fv\cdot\frac{1}{a}\nabla\left(\epsilon\partial_t\varphi\right)dxdvdt\\ &=\frac{1}{|\mathbb{S}^{d-1}|}\epsilon\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}}fv\cdot\frac{1}{a}\nabla\varphi dxdv-\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}}\left(f-\langle f\rangle\right)v\cdot\frac{1}{a}\nabla\left(\epsilon\partial_t\varphi\right)dxdv\end{aligned}$$

and

$$\begin{aligned}\epsilon\partial_t\varphi &=\left(-\tfrac{1}{\mathrm{d}}\nabla\cdot\left(\tfrac{1}{a}\nabla\right)\right)^{-1}\left(\epsilon\partial_t w_\epsilon\right)=\left(-\tfrac{1}{\mathrm{d}}\nabla\cdot\left(\tfrac{1}{a}\nabla\right)\right)^{-1}\left(-\langle v\cdot\nabla f\rangle-\epsilon\partial_t u\right)\\ &=\left(-\tfrac{1}{\mathrm{d}}\nabla\cdot\left(\tfrac{1}{a}\nabla\right)\right)^{-1}\langle -v\cdot\nabla\left(f-\langle f\rangle\right)\rangle+\epsilon u\ ,\end{aligned}$$

we see that

$$\begin{aligned}&\epsilon\int_\Omega\langle v\partial_t f\rangle\cdot\frac{1}{a}\nabla\varphi dx\\ &=\frac{1}{|\mathbb{S}^{d-1}|}\epsilon\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}}fv\cdot\frac{1}{a}\nabla\varphi dxdv\\ &\quad+\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}}\left(f-\langle f\rangle\right)v\cdot\frac{1}{a}\nabla\left(\left(-\frac{1}{\mathrm{d}}\nabla\cdot\left(\frac{1}{a}\nabla\right)\right)^{-1}\langle v\cdot\nabla\left(f-\langle f\rangle\right)\rangle\right)dxdv\\ &\quad-\epsilon\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}}\left(f-\langle f\rangle\right)v\cdot\frac{1}{a}\nabla udxdv\\ &\le\frac{1}{|\mathbb{S}^{d-1}|}\epsilon\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{d-1}}fv\cdot\frac{1}{a}\nabla\varphi dxdv+C\|f-\langle f\rangle\|^2_{L^2\left(\Omega\times\mathbb{S}^{d-1}\right)}+\epsilon^2C\|\nabla u\|^2_{L^2(\Omega)}\ .\end{aligned}$$

Combining the three above contributions with (3.2), one obtains

$$\begin{aligned}\frac{d}{dt}\int_{\Omega}\frac{1}{a}\left|\nabla\varphi\right|^{2}dx+\left\|\nabla\cdot\left(\frac{1}{a}\nabla\right)\varphi\right\|_{L^{2}(\Omega)}^{2}\quad&\leq\epsilon C\frac{d}{dt}\int_{\Omega\times\mathbb{S}^{\mathrm{d}-1}}fv\cdot\frac{1}{a}\nabla\varphi dxdv+\epsilon^{2}C\left\|\nabla u\right\|_{L^{2}(\Omega)}^{2}\\&+C\left(\left\|f\right\|_{L^{2}\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)}^{2}+\left\|f-\left\langle f\right\rangle\right\|_{L^{2}\left(\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)}^{2}\right)\ .\end{aligned}$$

Integrating the above over $(0,T)$, we observe with $\varphi=\left(-\frac{1}{\mathrm{d}}\nabla\cdot\left(\frac{1}{a}\nabla\right)\right)^{-1}w_{\epsilon}$ and $w_{\epsilon}=\left\langle f\right\rangle-u$ that

$$\begin{aligned}&\left\|w_{\epsilon}(\cdot,T)\right\|_{H^{-1}(\Omega)}^{2}+\left\|w_{\epsilon}\right\|_{L^{2}(\Omega\times(0,T))}^{2}\\\leq\ &\epsilon C\left(\left\|f_{|t=T}\right\|_{L^{2}\left(\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)}^{2}+\left\|u_{|t=T}\right\|_{L^{2}(\Omega)}^{2}+\left\|f_{0}\right\|_{L^{2}\left(\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)}^{2}+\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2}\right)\\&+\epsilon^{2}C\left\|\nabla u\right\|_{L^{2}(\Omega\times(0,T))}^{2}\\&+C\left(\left\|f\right\|_{L^{2}\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\times(0,T)\right)}^{2}+\left\|f-\left\langle f\right\rangle\right\|_{L^{2}\left(\Omega\times\mathbb{S}^{\mathrm{d}-1}\times(0,T)\right)}^{2}\right)\ .\end{aligned}$$

Next, we use the trace estimate in Proposition 3.2,

$$\int_{\Omega\times\mathbb{S}^{\mathrm{d}-1}}\left|f\left(x,v,T\right)\right|^{2}dxdv+\frac{2c_{min}}{\epsilon^{2}}\int_{0}^{T}\int_{\Omega\times\mathbb{S}^{\mathrm{d}-1}}\left|f-\left\langle f\right\rangle\right|^{2}dxdvdt\leq\int_{\Omega\times\mathbb{S}^{\mathrm{d}-1}}\left|f_{0}\right|^{2}dxdv$$

and

$$\int_{\Omega}\left|u\left(x,T\right)\right|^{2}dx+\frac{2}{\mathrm{d}c_{max}}\int_{0}^{T}\int_{\Omega}\left|\nabla u\right|^{2}dxdt\leq\int_{\Omega}\left|\left\langle f_{0}\right\rangle\right|^{2}dx$$

to get that

$$\begin{aligned}&\left\|\left(\left\langle f\right\rangle-u\right)(\cdot,T)\right\|_{H^{-1}(\Omega)}+\left\|\left\langle f\right\rangle-u\right\|_{L^{2}(\Omega\times(0,T))}\\\leq\ &\sqrt{\epsilon}C\left\|f_{0}\right\|_{L^{2}\left(\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)}+\epsilon^{\frac{1}{2p}}T^{\frac{p-1}{2p}}C_{p}C\|f_{0}\|_{L^{2p}(\Omega\times\mathbb{S}^{\mathrm{d}-1})}\ .\end{aligned}$$

This completes the proof.

4 Observation estimates for diffusion equation

In this section, we establish an observation estimate at one point in time for parabolic equations (see Theorem 4.1 below). Such estimate is an interpolation inequality. Hölder type inequalities of such form already appear in [LR] for elliptic operators by Carleman inequalities. It applies to the observability for the heat equation in manifold and to the sum of eigenfunctions estimate of Lebeau-Robbiano. On the other hand, for parabolic operators, Escauriaza, Fernandez and Vessella proved such interpolation estimate far from the boundary by some adequate Carleman estimates [EFV]. Here our approach is completely new and uses properties of the heat kernel with a parametrix of order 0.

Theorem 4.1 *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 1$, either convex or C^2 and connected. Let ω be a nonempty open subset of Ω, and $T > 0$. Let A be a $n \times n$ symmetric positive-definite matrix with $C^1\left(\overline{\Omega} \times [0,T]\right)$ coefficients such that $A(\cdot,T) \in C^2\left(\overline{\Omega}\right)$. There are $c > 0$ and $\mu \in (0,1)$ such that any solution to*

$$\begin{cases} \partial_t u - \nabla \cdot (A \nabla u) = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot,0) \in L^2(\Omega) \ , & \end{cases}$$

satisfies

$$\int_\Omega |u(x,T)|^2 \, dx \leq \left(c \int_\omega |u(x,T)|^2 \, dx \right)^{1-\mu} \left(\int_\Omega |u(x,0)|^2 \, dx \right)^{\mu} \ .$$

Moreover, when A is time-independent, then $c = Ce^{\frac{C}{T}}$ where C and μ only depend on (A, Ω, ω, n).

Clearly, Proposition 2.1 is a direct application of Theorem 4.1. The proof of Theorem 4.1 uses covering argument and propagation of interpolation inequalities along a chain of balls (also called propagation of smallness): First we extend $A(\cdot,T)$ to a C^2 function on $\mathbb{R}^n$ denoted A_T. Next, for each $x_0 \in \mathbb{R}^n$ there are a neighborhood of x_0 and a function $x \mapsto d(x,x_0)$ on which the following four properties hold:

1. $\frac{1}{C} |x - x_0| \leq d(x,x_0) \leq C |x - x_0|$ for some $C \geq 1$ depending on (x_0, A_T) ;
2. $x \mapsto d^2(x,x_0)$ is C^2 ;
3. $A_T(x) \nabla d(x,x_0) \cdot \nabla d(x,x_0) = 1$;
4. $\frac{1}{2} A_T(x) \nabla^2 d^2(x,x_0) = I_n + O(d(x,x_0))$.

Here ∇^2 denotes the Hessian matrix and $d(x,x_0)$ is the geodesic distance connecting x to x_0. The proof of the above properties for $d(x,x_0)$ is a consequence of Gauss's lemma for C^2 metrics (see [IM, page 7]).

Now we are able to define the ball of center x_0 and radius R as $B_R = \{x; d(x,x_0) < R\}$. We will choose $x_0 \in \Omega$ in order that one of the two following assumptions hold: (i) $\overline{B_r} \subset \Omega$ for any $r \in (0,R]$; (ii) $B_r \cap \partial\Omega \neq \emptyset$ and $A \nabla d^2 \cdot \nu \geq 0$ on $\partial\Omega \cap B_R$ for any $r \in [R_0, R]$ where $R_0 > 0$. Here ν is the unit outward normal vector to $\partial\Omega \cap B_R$.

The case (i) deals with the propagation in the interior domain by a chain of balls strictly included in Ω. The analysis near the boundary $\partial\Omega$ requires the assumptions of (ii).

However when $\Omega \subset \mathbb{R}^n$ is a convex domain or a star-shaped domain with respect to $x_0 \in \Omega$, we only need to propagate the estimate in the interior domain.

If further $A = I_n$, then $d(x, x_0) = |x - x_0|$ and it is well defined for any $x \in \Omega$. From [PW] such observation at one point in time implies the observability for the heat equation which from [AEWZ] is equivalent to the sum of eigenfunctions estimate of Lebeau-Robbiano type. Eventually a careful evaluation of the constants gives the following estimates (whose proof is omitted).

Theorem 4.2 *Suppose that $\Omega \subset \mathbb{R}^n$ is a convex domain or a star-shaped domain with respect to $x_0 \in \Omega$ such that $\{x; |x - x_0| < r\} \Subset \Omega$ for some $r > 0$. Then for any $u_0 \in L^2(\Omega), T > 0$, $(a_i)_{i \geq 1} \in \mathbb{R}$, $\mu \geq 1$, $\varepsilon \in (0, 1)$, one has*

$$\left\| e^{T\Delta} u_0 \right\|_{L^2(\Omega)} \leq \frac{1}{r^n} \frac{1}{r^{\varepsilon(n-2)}} e^{\frac{C}{T} \frac{1}{r^{6\varepsilon}}} \int_0^T \left\| e^{t\Delta} u_0 \right\|_{L^2(|x - x_0| < r)} dt$$

and

$$\sum_{\mu_i \leq \mu} |a_i|^2 \leq \frac{1}{r^{2n(1+\varepsilon)}} e^{C \frac{1}{r^{2\varepsilon}} \sqrt{\mu}} \int_{|x - x_0| < r} \left| \sum_{\mu_i \leq \mu} a_i e_i(x) \right|^2 dx$$

where $C > 0$ is a constant only depending on $\left(\varepsilon, n, \max\left\{|x - x_0|; x \in \overline{\Omega}\right\}\right)$. Here (μ_i, e_i) denotes the eigenbasis of the Laplace operator with Dirichlet boundary condition.

In the next subsection, we state some preliminary lemmas and corollaries. In subsection 4.2, we prove Theorem 4.1. Subsection 4.3 is devoted to the proof of the preliminary results.

4.1 Preliminary results

In this subsection we present some lemmas and corollaries which will be used for the proof of Theorem 4.1.

The following lemma allows to solve differential inequalities and makes appear the Hölder type of inequality in Theorem 4.1.

Lemma 4.3 *Let $T > 0$, $\lambda > 0$ and $F_1, F_2 \in C^0([0, T])$. Consider two positive functions $y, N \in C^1([0, T])$ such that*

$$\begin{cases} \left| \frac{1}{2} y'(t) + N(t) y(t) \right| \leq \left(\frac{C_0}{T - t + \lambda} + C_1 \right) y(t) + F_1(t) y(t) \\ N'(t) \leq \left(\frac{1 + C_0}{T - t + \lambda} + C_1 \right) N(t) + F_2(t) \end{cases}$$

where $C_0, C_1 \geq 0$. *Then for any* $0 \leq t_1 < t_2 < t_3 \leq T$, *one has*

$$y\left(t_2\right)^{1+M} \leq y\left(t_3\right) y\left(t_1\right)^M e^{4D} \left(\frac{T-t_1+\lambda}{T-t_3+\lambda}\right)^{2C_0(1+M)}$$

where

$$M = \frac{\displaystyle\int_{t_2}^{t_3} \frac{e^{tC_1}}{\left(T-t+\lambda\right)^{1+C_0}} dt}{\displaystyle\int_{t_1}^{t_2} \frac{e^{tC_1}}{\left(T-t+\lambda\right)^{1+C_0}} dt} \text{ and } D = M\left(t_2-t_1\right)\left(C_1 + \sup_{[t_1,t_3]} \left|F_1\right| + \int_{t_1}^{t_3} \left|F_2\right| dt\right) .$$

Corollary 4.4 *Under the assumptions of Lemma 1, for any* $\lambda > 0$ *and* $\ell > 1$ *such that* $\ell\lambda < T/4$, *one has*

$$y\left(T-\ell\lambda\right)^{1+M_\ell} \leq y\left(T\right) y\left(T-2\ell\lambda\right)^{M_\ell} e^{D_\ell} \left(2\ell+1\right)^{2C_0(1+M_\ell)}$$

where $D_\ell = TM_\ell\left(C_1 + \sup_{[t_1,t_3]}\left|F_1\right| + \int_{t_1}^{t_3}\left|F_2\right| dt\right)$, $M_\ell \leq e^{C_1 T} \frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}}$ *if* $C_0 > 0$ *and* $M_\ell \leq e^{C_1 T}\frac{\ln(\ell+1)}{\ln 2}$ *if* $C_0 = 0$.

Proof .- Apply Lemma 4.3 with $t_3 = T$, $t_2 = T-\ell\lambda$, $t_1 = T-2\ell\lambda$, with $\ell\lambda < T/4$. Here when $C_0 > 0$

$$M_\ell = \frac{\displaystyle\int_{T-\ell\lambda}^{T} \frac{e^{tC_1}}{\left(T-t+\lambda\right)^{1+C_0}} dt}{\displaystyle\int_{T-2\ell\lambda}^{T-\ell\lambda} \frac{e^{tC_1}}{\left(T-t+\lambda\right)^{1+C_0}} dt} \leq e^{2\ell\lambda C_1} \frac{\left(\ell+1\right)^{C_0} - 1}{1-\left(\frac{\ell+1}{2\ell+1}\right)^{C_0}} \leq e^{C_1 T} \frac{\left(\ell+1\right)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}} \text{ for } \ell > 1 .$$

And when $C_0 = 0$

$$M_\ell = \frac{\displaystyle\int_{T-\ell\lambda}^{T} \frac{e^{tC_1}}{\left(T-t+\lambda\right)} dt}{\displaystyle\int_{T-2\ell\lambda}^{T-\ell\lambda} \frac{e^{tC_1}}{\left(T-t+\lambda\right)} dt} \leq e^{2\ell\lambda C_1} \frac{\ln\left(\ell+1\right)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)} \leq e^{C_1 T} \frac{\ln\left(\ell+1\right)}{\ln 2} \text{ for } \ell > 1 .$$

The following lemma establishes the differential inequalities associated to parabolic equations in any open set $\vartheta \subset \mathbb{R}^n$:

Lemma 4.5 *For any $\xi \in C^2\left(\overline{\Omega}\times[0,T]\right)$, $z \in H^1\left(0,T;H_0^1\left(\vartheta\right)\right)$, one has*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\vartheta |z|^2 e^\xi dx + \int_\vartheta A\nabla z\cdot\nabla z e^\xi dx \\
&= \frac{1}{2}\int_\vartheta |z|^2\left(\partial_t\xi + \nabla\cdot\left(A\nabla\xi\right) + A\nabla\xi\cdot\nabla\xi\right) e^\xi dx + \int_\vartheta z\left(\partial_t z - \nabla\cdot\left(A\nabla z\right)\right) e^\xi dx
\end{aligned}$$

and for some C only depending on $(A,\partial_x A,\partial_t A)$

$$\begin{aligned}
\frac{d}{dt}\frac{\displaystyle\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx} \leq\ & \frac{-2\displaystyle\int_\vartheta A\nabla^2\xi A\nabla z\cdot\nabla z e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx} + \frac{\displaystyle\int_{\partial\vartheta}\left(A\nabla z\cdot\nabla z\right)\left(A\nabla\xi\cdot\nu\right) e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx} \\
&+\frac{\displaystyle\int_\vartheta \left|\partial_t z - \nabla\cdot\left(A\nabla z\right)\right|^2 e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx} + C\frac{\displaystyle\int_\vartheta \left(1+|\nabla\xi|\right)|\nabla z|^2 e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx} \\
&+\frac{\displaystyle\int_\vartheta A\nabla z\cdot\nabla z\left(\partial_t\xi + \nabla\cdot\left(A\nabla\xi\right) + A\nabla\xi\cdot\nabla\xi\right) e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx} \\
&-\frac{\displaystyle\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx}\times\frac{\displaystyle\int_\vartheta |z|^2\left(\partial_t\xi + \nabla\cdot\left(A\nabla\xi\right) + A\nabla\xi\cdot\nabla\xi\right) e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx}\ .
\end{aligned}$$

Corollary 4.6 *Let $R>0$ be sufficiently small and $z \in H^1\left(0,T;H_0^1\left(\Omega\cap B_R\right)\right)$ with $B_R = \{x; d\left(x,x_0\right) < R\}$. Introduce for $t \in (0,T]$, $\mathcal{P}z = \partial_t z - \nabla\cdot\left(A\nabla z\right)$,*

$$G_\lambda\left(x,t\right) = \frac{1}{\left(T-t+\lambda\right)^{n/2}} e^{-\frac{d^2(x,x_0)}{4(T-t+\lambda)}} \quad \forall x \in B_R\ ,$$

and

$$N_\lambda\left(t\right) = \frac{\displaystyle\int_{\Omega\cap B_R} A\left(x,t\right)\nabla z\left(x,t\right)\cdot\nabla z\left(x,t\right) G_\lambda\left(x,t\right) dx}{\displaystyle\int_{\Omega\cap B_R} \left|z\left(x,t\right)\right|^2 G_\lambda\left(x,t\right) dx}$$

whenever $\displaystyle\int_{\Omega\cap B_R} \left|z\left(x,t\right)\right|^2 dx \neq 0$. Then, the following two properties hold:

i) For some $C_0 \geq 0$,

$$\begin{aligned}
&\left|\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_R} \left|z\left(x,t\right)\right|^2 G_\lambda\left(x,t\right) dx + N_\lambda\left(t\right)\int_{\Omega\cap B_R} \left|z\left(x,t\right)\right|^2 G_\lambda\left(x,t\right) dx\right| \\
&\leq \left(\frac{C_0}{T-t+\lambda} + C_1\right)\int_{\Omega\cap B_R} \left|z\left(x,t\right)\right|^2 G_\lambda\left(x,t\right) dx \\
&\quad + \int_{\Omega\cap B_R} \left|z\left(x,t\right)\mathcal{P}z\left(x,t\right)\right| G_\lambda\left(x,t\right) dx\ .
\end{aligned}$$

ii) There are $R>0$, $0\leq C_0<1$, $C_1\geq 0$ such that when $A\nabla d^2\cdot\nu\geq 0$ on $\partial\Omega\cap B_R$,

$$\frac{d}{dt}N_\lambda(t)\leq\left(\frac{1+C_0}{T-t+\lambda}+C_1\right)N_\lambda(t)+\frac{\displaystyle\int_{\Omega\cap B_R}\left|\mathcal{P}z(x,t)\right|^2G_\lambda(x,t)\,dx}{\displaystyle\int_{\Omega\cap B_R}\left|z(x,t)\right|^2G_\lambda(x,t)\,dx}\ .$$

Proof .- Apply Lemma 4.5 with $\vartheta=\Omega\cap B_R$ and

$$\xi(x,t)=-\frac{d^2(x,x_0)}{4(T-t+\lambda)}-\frac{n}{2}\ln(T-t+\lambda)\ .$$

It remains to bound $\partial_t\xi+\nabla\cdot(A\nabla\xi)+A\nabla\xi\cdot\nabla\xi$, $-2A\nabla^2\xi A\nabla z\cdot\nabla z$ and $|\nabla\xi|$. First, one get

$$C_{A_T}\left|\nabla\xi\right|^2\leq A_T\nabla\xi\cdot\nabla\xi=\frac{d^2(x,x_0)}{4(T-t+\lambda)^2}\ .$$

Next, $\nabla\cdot(A_T\nabla\xi)=\frac{-n}{(T-t+\lambda)}+\frac{O(d(x,x_0))}{T-t+\lambda}$ and

$$\partial_t\xi+\nabla\cdot(A_T\nabla\xi)+A_T\nabla\xi\cdot\nabla\xi=\frac{O(d(x,x_0))}{T-t+\lambda}$$

imply

$$\begin{array}{ll}&\partial_t\xi+\nabla\cdot(A\nabla\xi)+A\nabla\xi\cdot\nabla\xi\\=&\frac{O(d(x,x_0))}{T-t+\lambda}+\nabla\cdot\left((A-A_T)(A_T)^{-1}A_T\nabla\xi\right)+(A-A_T)\nabla\xi\cdot\nabla\xi\\=&\frac{O(d(x,x_0))}{T-t+\lambda}+O(1)\ ,\end{array}$$

where in the last equality we used $\left\|A(\cdot,t)-A_T\right\|\leq\left\|\partial_tA\right\|(T-t+\lambda)$. Finally, we have

$$\begin{array}{rl}-2A\nabla^2\xi A\nabla z\cdot\nabla z&=\frac{1}{2(T-t+\lambda)}A_T\nabla^2d^2A\nabla z\cdot\nabla z+(A-A_T)\nabla^2d^2A\nabla z\cdot\nabla z\\&=A\nabla z\cdot\nabla z\left(\frac{1+O(d(x,x_0))}{T-t+\lambda}+O(1)\right)\ .\end{array}$$

One conclude by choosing $R>0$ sufficiently small in order the constant C_0 in Corollary 4.6 satisfies $0<C_0<1$.

Remark .- When A is time-independent, then $C_1=0$ in Corollary 4.6.

The following lemma will be used to deal with the delocalized terms.

Lemma 4.7 *Let $\rho\in(0,R)$ and $0<\varepsilon<\rho/2$. There are constants $c_1,c_2,c_3>0$ only depending on (ρ,ε,A) such that for any $T-\theta\leq t\leq T$, one has*

$$\frac{\displaystyle\int_\Omega\left|u(x,0)\right|^2dx}{\displaystyle\int_{\Omega\cap B_\rho}\left|u(x,t)\right|^2dx}\leq e^{\frac{c_1}{\theta}}$$

where

$$\frac{1}{\theta}=c_2\ln\left(e^{c_3\left(1+\frac{1}{T}\right)}\frac{\displaystyle\int_\Omega |u(x,0)|^2\,dx}{\displaystyle\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2\,dx}\right)\ \textit{with}\ 0<\theta\le\min\left(1,T/2\right)\ .$$

The interested reader may wish here to compare this lemma with [EFV, Lemma 5].

4.2 Proof of Theorem 4.1

Let $\lambda>0$ and $\ell>1$ be such that $\ell\lambda<T/4$. By Corollary 4.4 with $y(t)=\displaystyle\int_{\Omega\cap B_R}|z(x,t)|^2\,G_\lambda(x,t)\,dx$, $N(t)=N_\lambda(t)$ given in Corollary 4.6,

$$F_1(t)=\frac{\displaystyle\int_{\Omega\cap B_R}|z(x,t)\left(\partial_t z(x,t)-\nabla\cdot\left(A(x,t)\nabla z(x,t)\right)\right)|\,G_\lambda(x,t)\,dx}{\displaystyle\int_{\Omega\cap B_R}|z(x,t)|^2\,G_\lambda(x,t)\,dx}$$

and

$$F_2(t)=\frac{\displaystyle\int_{\Omega\cap B_R}|\partial_t z(x,t)-\nabla\cdot\left(A(x,t)\nabla z(x,t)\right)|^2\,G_\lambda(x,t)\,dx}{\displaystyle\int_{\Omega\cap B_R}|z(x,t)|^2\,G_\lambda(x,t)\,dx}$$

knowing that $N'(t)\le\left(\frac{1+C_0}{T-t+\lambda}+C_1\right)N(t)+F_2(t)$ from Corollary 4.6, one can deduce the following interpolation inequality with $M_\ell\le e^{C_1T}\frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}}$ and $0<C_0<1$,

$$y(T-\ell\lambda)^{1+M_\ell}\le y(T)\,y(T-2\ell\lambda)^{M_\ell}\,(2\ell+1)^{2C_0(1+M_\ell)}\,e^{D_\ell}$$

that is

$$\begin{aligned}&\left(\int_{\Omega\cap B_R}|z(x,T-\ell\lambda)|^2\,e^{\frac{-d^2(x,x_0)}{4(\ell+1)\lambda}}\,dx\right)^{1+M_\ell}\\ &\le(\ell+1)^{n/2}\,(2\ell+1)^{2C_0(1+M_\ell)}\int_{\Omega\cap B_R}|z(x,T)|^2\,e^{\frac{-d^2(x,x_0)}{4\lambda}}\,dx\left(\int_\Omega|u(x,0)|^2\,dx\right)^{M_\ell}\\ &\quad\times e^{TM_\ell\left(\int_{T-2\ell\lambda}^{T}|F_2|\,dt+\sup_{[T-2\ell\lambda,T]}|F_1|+C_1\right)}\ .\end{aligned}$$

From the definition of F_1,

$$\left|F_1(t)\right| \leq e^{\frac{(R-2\varepsilon)^2}{4(T-t+\lambda)}} e^{-\frac{(R-\varepsilon)^2}{4(T-t+\lambda)}} \frac{\displaystyle\int_{\Omega\cap\{R-\varepsilon\leq d(x,x_0)\}} |\chi u| \left|-2A\nabla\chi\cdot\nabla u - \nabla\cdot(A\nabla\chi)\,u\right| dx}{\displaystyle\int_{\Omega\cap B_{R-2\varepsilon}} |u|^2\, dx} .$$

Since $e^{\frac{(R-2\varepsilon)^2}{4(T-t+\lambda)}} e^{-\frac{(R-\varepsilon)^2}{4(T-t+\lambda)}} = e^{-\frac{\varepsilon(2R-3\varepsilon)}{4(T-t+\lambda)}} \leq e^{-\frac{\varepsilon(2R-3\varepsilon)}{12\ell\lambda}}$ for $t\in[T-2\ell\lambda,T]$ with $\ell>1$, one has when $t\in[T-2\ell\lambda,T]$

$$\left|F_1(t)\right| \leq e^{-\frac{\varepsilon(2R-3\varepsilon)}{12\ell\lambda}} \frac{\displaystyle\int_{\Omega\cap\{R-\varepsilon\leq d(x,x_0)\}} |\chi u| \left|-2A\nabla\chi\cdot\nabla u - \nabla\cdot(A\nabla\chi)\,u\right| dx}{\displaystyle\int_{\Omega\cap B_{R-2\varepsilon}} |u|^2\, dx} .$$

By Lemma 4.7 with $\rho = R-2\varepsilon$,

$$\sup_{t\in[T-2\ell\lambda,T]} \left|F_1(t)\right| \leq e^{-\frac{\varepsilon(2R-3\varepsilon)}{12\ell\lambda}} c e^{\frac{c_1}{\theta}} \text{ if } 2\ell\lambda\leq\theta .$$

Similarly, from the definition of F_2,

$$\left|F_2(t)\right| \leq e^{\frac{(R-2\varepsilon)^2}{4(T-t+\lambda)}} e^{-\frac{(R-\varepsilon)^2}{4(T-t+\lambda)}} \frac{\displaystyle\int_{\Omega\cap\{R-\varepsilon\leq d(x,x_0)\}} \left|-2A\nabla\chi\cdot\nabla u - \nabla\cdot(A\nabla\chi)\,u\right|^2 dx}{\displaystyle\int_{\Omega\cap B_{R-2\varepsilon}} |u|^2\, dx}$$

and then, when $t\in[T-2\ell\lambda,T]$

$$\left|F_2(t)\right| \leq e^{-\frac{\varepsilon(2R-3\varepsilon)}{12\ell\lambda}} \frac{\displaystyle\int_{\Omega\cap\{R-\varepsilon\leq d(x,x_0)\}} \left|-2A\nabla\chi\cdot\nabla u - \nabla\cdot(A\nabla\chi)\,u\right|^2 dx}{\displaystyle\int_{\Omega\cap B_{R-2\varepsilon}} |u|^2\, dx} .$$

By Lemma 4.7 with $\rho = R-2\varepsilon$,

$$\int_{T-2\ell\lambda}^{T} \left|F_2(t)\right| dt \leq e^{-\frac{\varepsilon(2R-3\varepsilon)}{12\ell\lambda}} c e^{\frac{c_1}{\theta}} \text{ if } 2\ell\lambda\leq\theta$$

where $c>1$ is a constant only dependent on (A,R,ε). We conclude that for any $2\ell\lambda \leq \theta\frac{\varepsilon(2R-3\varepsilon)}{6c_1}$

$$\sup_{t\in[T-\theta,T]} \left|F_1(t)\right| + \int_{T-\theta}^{T} \left|F_2(t)\right| dt \leq 2c .$$

Therefore there is $c_4 := \frac{\varepsilon(2R-3\varepsilon)}{6c_1} \in (0,1)$ such that for any $2\ell\lambda\leq c_4\theta$

$$\begin{aligned}
&\left(\int_{\Omega\cap B_R} \left|z(x,T-\ell\lambda)\right|^2 e^{\frac{-d^2(x,x_0)}{4(\ell+1)\lambda}} dx\right)^{1+M_\ell} \\
&\leq e^{(2c+C_1)TM_\ell} (2\ell+1)^{2C_0(1+M_\ell)+n/2} \int_{\Omega\cap B_R} \left|z(x,T)\right|^2 e^{\frac{-d^2(x,x_0)}{4\lambda}} dx \left(\int_\Omega \left|u(x,0)\right|^2 dx\right)^{M_\ell}
\end{aligned}$$

which implies

$$\left(\int_{\Omega\cap B_R}\left|z\left(x,T-\ell\lambda\right)\right|^2dx\right)^{1+M_\ell}\leq e^{(2c+C_1)TM_\ell}\left(2\ell+1\right)^{2C_0(1+M_\ell)+n/2}e^{\frac{R^2}{4(\ell+1)\lambda}(1+M_\ell)}$$
$$\times\int_{\Omega\cap B_R}\left|z\left(x,T\right)\right|^2e^{\frac{-d^2(x,x_0)}{4\lambda}}dx\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^{M_\ell}\ .$$

Now, we split $\int_{\Omega\cap B_R}\left|z\left(x,T\right)\right|^2e^{\frac{-d^2(x,x_0)}{4\lambda}}dx$ into two parts: For any $0<r<R/2$ such that $B_r\Subset\Omega$,

$$\int_{\Omega\cap B_R}\left|z\left(x,T\right)\right|^2e^{\frac{-d^2(x,x_0)}{4\lambda}}dx\leq\int_{B_r}\left|u\left(x,T\right)\right|^2dx+e^{\frac{-r^2}{4\lambda}}\int_\Omega\left|u\left(x,0\right)\right|^2dx\ .$$

Consequently, we have

$$\left(\int_{\Omega\cap B_R}\left|z\left(x,T-\ell\lambda\right)\right|^2dx\right)^{1+M_\ell}$$
$$\leq e^{(2c+C_1)TM_\ell}\left(2\ell+1\right)^{2C_0(1+M_\ell)+n/2}e^{\frac{R^2}{4(\ell+1)\lambda}(1+M_\ell)}\int_{B_r}\left|u\left(x,T\right)\right|^2dx\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^{M_\ell}$$
$$+e^{(2c+C_1)TM_\ell}\left(2\ell+1\right)^{2C_0(1+M_\ell)+n/2}e^{\frac{R^2}{4(\ell+1)\lambda}(1+M_\ell)}e^{\frac{-r^2}{4\lambda}}\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^{1+M_\ell}\ .$$

Now, choose $\ell>1$ in order that $\frac{R^2}{4(\ell+1)}\left(1+M_\ell\right)\leq\frac{r^2}{8}$ (knowing that $M_\ell\leq e^{C_1T}\frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}}$ for $\ell>1$ and $C_0<1$). Therefore there is $K>1$ such that for any $\lambda\leq\frac{c_4}{2\ell}\theta$

$$\left(\int_{\Omega\cap B_{R-\varepsilon}}\left|u\left(x,T-\ell\lambda\right)\right|^2dx\right)^{1+K}\leq Ke^{\frac{r^2}{8\lambda}}\int_{B_r}\left|u\left(x,T\right)\right|^2dx\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^K$$
$$+Ke^{\frac{-r^2}{8\lambda}}\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^{1+K}\ .$$

But by Lemma 4.7 with $\rho=R-2\varepsilon$, since $\ell\lambda\leq\theta$,

$$\int_\Omega\left|u\left(x,T\right)\right|^2dx\leq\int_\Omega\left|u\left(x,0\right)\right|^2dx\leq e^{\frac{c_1}{\theta}}\int_{\Omega\cap B_{R-\varepsilon}}\left|u\left(x,T-\ell\lambda\right)\right|^2dx\ .$$

As a consequence, for any $\lambda\leq\frac{c_4}{2\ell}\theta$ one obtain

$$\left(\int_\Omega\left|u\left(x,T\right)\right|^2dx\right)^{1+K}\leq e^{\frac{(1+K)c_1}{\theta}}Ke^{\frac{r^2}{8\lambda}}\int_{B_r}\left|u\left(x,T\right)\right|^2dx\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^K$$
$$+e^{\frac{(1+K)c_1}{\theta}}Ke^{\frac{-r^2}{8\lambda}}\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^{1+K}\ .$$

On the other hand, for any $\lambda\in\left(\frac{c_4}{2\ell}\theta,\frac{T}{4\ell}\right)$, one has $1\leq e^{\frac{-r^2}{8\lambda}}e^{\frac{r^2\ell}{4c_4\theta}}$. And for any $\lambda\geq\frac{T}{4\ell}$, there holds $1\leq e^{\frac{-r^2}{8\lambda}}e^{\frac{r^2\ell}{4T}}$. Finally, there is $K>1$ such that for any $\lambda>0$,

$$\left(\int_\Omega\left|u\left(x,T\right)\right|^2dx\right)^{1+K}\leq e^{\frac{K}{\theta}}Ke^{\frac{r^2}{8\lambda}}\int_{B_r}\left|u\left(x,T\right)\right|^2dx\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^K$$
$$+e^{\frac{K}{\theta}}Ke^{\frac{K}{T}}e^{\frac{-r^2}{8\lambda}}\left(\int_\Omega\left|u\left(x,0\right)\right|^2dx\right)^{1+K}\ .$$

Next, choose $\lambda > 0$ such that $e^{\frac{r^2}{8\lambda}} := 2e^{\frac{K}{\theta}} K e^{\frac{K}{T}} \left(\frac{\displaystyle\int_\Omega |u(x,0)|^2 dx}{\displaystyle\int_\Omega |u(x,T)|^2 dx} \right)^{1+K}$ that is

$$e^{\frac{K}{\theta}} K e^{\frac{K}{T}} e^{\frac{-r^2}{8\lambda}} \left(\int_\Omega |u(x,0)|^2 dx \right)^{1+K} = \frac{1}{2} \left(\int_\Omega |u(x,T)|^2 dx \right)^{1+K}$$

in order that

$$\int_\Omega |u(x,T)|^2 dx \leq 2Ke^{\frac{K}{\theta}} \left(e^{\frac{K}{T}} \int_{B_r} |u(x,T)|^2 dx \right)^{\frac{1}{2+2K}} \left(\int_\Omega |u(x,0)|^2 dx \right)^{\frac{1+2K}{2+2K}} .$$

Recall that by Lemma 4.7 with $\rho = R - 2\varepsilon$,

$$e^{\frac{K}{\theta}} = \left(e^{c_3\left(1+\frac{1}{T}\right)} \frac{\displaystyle\int_\Omega |u(x,0)|^2 dx}{\displaystyle\int_{\Omega\cap B_{R-4\varepsilon}} |u(x,T)|^2 dx} \right)^{Kc_2} .$$

Finally, we obtain

$$\int_{\Omega\cap B_{R-4\varepsilon}} |u(x,T)|^2 dx \leq K \left(\int_\Omega |u(x,0)|^2 dx \right)^{\frac{K}{1+K}} \left(e^{\frac{K}{T}} \int_{B_r} |u(x,T)|^2 dx \right)^{\frac{1}{1+K}}$$

for some positive constant K only depending on $(A_T, \varepsilon, R, r, n)$. By an adequate covering of Ω by balls $B_{R-4\varepsilon}$ where x_0 and R are chosen such that $A\nabla d^2 \cdot \nu \geq 0$ on $\partial\Omega \cap B_R$ and by a propagation of smallness based on the previous estimate, we get the desired observation inequality at one point in time for parabolic equations.

4.3 Proof of Lemma 4.3

We shall distinguish two cases: $t \in [t_1, t_2]$; $t \in [t_2, t_3]$. For $t_1 \leq t \leq t_2$, we integrate $\left((T-t+\lambda)^{1+C_0} e^{-tC_1} N(t) \right)' \leq (T-t+\lambda)^{1+C_0} e^{-tC_1} F_2(t)$ over (t, t_2) to get

$$\left(\frac{T-t_2+\lambda}{T-t+\lambda} \right)^{1+C_0} e^{-C_1(t_2-t)} N(t_2) - \int_{t_1}^{t_2} |F_2(s)| \, ds \leq N(t) .$$

Then we solve $y' + 2\alpha(t) y \leq 0$ with

$$\alpha(t) = \left(\frac{T-t_2+\lambda}{T-t+\lambda} \right)^{1+C_0} e^{-C_1(t_2-t)} N(t_2) - \frac{C_0}{T-t+\lambda} - C_1 - \int_{t_1}^{t_2} |F_2| \, ds - \sup_{[t_1,t_2]} |F_1|$$

and integrate it over (t_1, t_2) to obtain

$$\begin{aligned} & e^{2N(t_2)\int_{t_1}^{t_2}\left(\frac{T-t_2+\lambda}{T-t+\lambda}\right)^{1+C_0} e^{-C_1(t_2-t)}dt} \\ & \leq \frac{y(t_1)}{y(t_2)}\left(\frac{T-t_1+\lambda}{T-t_2+\lambda}\right)^{2C_0} e^{2(t_2-t_1)\left(C_1+\int_{t_1}^{t_2}|F_2|\,ds+\sup_{[t_1,t_2]}|F_1|\right)} . \end{aligned}$$

For $t_2 \leq t \leq t_3$, we integrate $\left((T-t+\lambda)^{1+C_0} e^{-tC_1} N(t)\right)' \leq (T-t+\lambda)^{1+C_0} F_2(t)$ over (t_2, t) to get

$$N(t) \leq e^{C_1(t-t_2)}\left(\frac{T-t_2+\lambda}{T-t+\lambda}\right)^{1+C_0}\left(N(t_2)+\int_{t_2}^{t_3}|F_2(s)|\,ds\right) .$$

Then we solve $0 \leq y' + 2\alpha(t)\, y$ with

$$\alpha(t) = \left(\frac{T-t_2+\lambda}{T-t+\lambda}\right)^{1+C_0} e^{C_1(t-t_2)}\Big(N(t_2)+\int_{t_2}^{t_3}|F_2|\,ds+\sup_{[t_2,t_3]}|F_1|+C_1\Big)+\frac{C_0}{T-t+\lambda}$$

and integrate it over (t_2, t_3) to obtain

$$\begin{aligned} y(t_2) & \leq e^{2\left(N(t_2)+\int_{t_2}^{t_3}|F_2|\,ds+\sup_{[t_2,t_3]}|F_1|+C_1\right)\int_{t_2}^{t_3}\left(\frac{T-t_2+\lambda}{T-t+\lambda}\right)^{1+C_0} e^{C_1(t-t_2)}dt} \\ & \quad \times y(t_3)\left(\frac{T-t_2+\lambda}{T-t_3+\lambda}\right)^{2C_0} . \end{aligned}$$

Finally, combining the case $t_1 \leq t \leq t_2$ and the case $t_2 \leq t \leq t_3$, we have

$$\begin{aligned} y(t_2) & \leq y(t_3)\left(\frac{y(t_1)}{y(t_2)}\right)^{M}\left(\frac{T-t_2+\lambda}{T-t_3+\lambda}\right)^{2C_0}\left(\frac{T-t_1+\lambda}{T-t_2+\lambda}\right)^{2C_0 M} \\ & \quad \times e^{2M(t_2-t_1)\int_{t_1}^{t_2}|F_2|\,ds}\, e^{2M(t_2-t_1)\int_{t_2}^{t_3}|F_2|\,ds} \\ & \quad \times e^{2M(t_2-t_1)\left(\sup_{[t_1,t_2]}|F_1|+C_1\right)}\, e^{2M(t_2-t_1)\left(\sup_{[t_2,t_3]}|F_1|+C_1\right)} \end{aligned}$$

with

$$M = \frac{\displaystyle\int_{t_2}^{t_3}\frac{e^{tC_1}}{(T-t+\lambda)^{1+C_0}}dt}{\displaystyle\int_{t_1}^{t_2}\frac{e^{tC_1}}{(T-t+\lambda)^{1+C_0}}dt}$$

which is the desired estimate.

4.4 Proof of Lemma 4.5

The aim of this section is to prove the differential inequalities for parabolic equations stated in Lemma 4.5. For any $z \in H^1(0,T;H_0^1(\vartheta))$, a weak solution of $\partial_t z - \nabla \cdot (A\nabla z) = g$ with $g \in L^2(\Omega \times (0,T))$, we apply the following formula

$$\int_\vartheta \partial_t z\varphi dx + \int_\vartheta A\nabla z \cdot \nabla\varphi dx = \int_{\partial\vartheta} A\nabla z \cdot \nu\varphi dx + \int_\vartheta g\varphi dx$$

with different functions φ: $\varphi = ze^\xi$, $\varphi = \partial_t ze^\xi$ and $\varphi = A\nabla z \cdot \nabla\xi e^\xi$. Here ν is the unit outward normal vector to $\partial\vartheta$ and $\xi = \xi(x,t)$ is a sufficiently smooth function which will be chosen later. When $\varphi = ze^\xi$, we have

$$\int_\vartheta A\nabla z \cdot \nabla ze^\xi dx = -\int_\vartheta \left(\partial_t z + A\nabla z \cdot \nabla\xi - \frac{1}{2}g\right) ze^\xi dx + \frac{1}{2}\int_\vartheta gze^\xi dx$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\vartheta |z|^2 e^\xi dx = \int_\vartheta z\partial_t ze^\xi dx + \frac{1}{2}\int_\vartheta |z|^2 \partial_t\xi e^\xi dx \\
&= -\int_\vartheta A\nabla z \cdot \nabla ze^\xi dx - \int_\vartheta A\nabla z \cdot \nabla\xi ze^\xi dx + \int_\vartheta gze^\xi dx + \frac{1}{2}\int_\vartheta |z|^2 \partial_t\xi e^\xi dx \\
&= -\int_\vartheta A\nabla z \cdot \nabla ze^\xi dx - \frac{1}{2}\int_\vartheta A\nabla\left(z^2\right) \cdot \nabla\xi e^\xi dx + \int_\vartheta gze^\xi dx + \frac{1}{2}\int_\vartheta |z|^2 \partial_t\xi e^\xi dx \\
&= -\int_\vartheta A\nabla z \cdot \nabla ze^\xi dx + \frac{1}{2}\int_\vartheta |z|^2 \left(\partial_t\xi + \nabla \cdot (A\nabla\xi) + A\nabla\xi \cdot \nabla\xi\right) e^\xi dx + \int_\vartheta gze^\xi dx\ .
\end{aligned}$$

When $\varphi = \partial_t ze^\xi$, we have

$$\int_\vartheta |\partial_t z|^2 e^\xi dx + \int_\vartheta A\nabla z \cdot \partial_t\nabla ze^\xi dx + \int_\vartheta A\nabla z \cdot \nabla\xi\partial_t ze^\xi dx = \int_\vartheta g\partial_t ze^\xi dx$$

which implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\vartheta A\nabla z \cdot \nabla ze^\xi dx - \frac{1}{2}\int_\vartheta \partial_t A\nabla z \cdot \nabla ze^\xi dx \\
&= \int_\vartheta A\nabla z \cdot \partial_t\nabla ze^\xi dx + \frac{1}{2}\int_\vartheta A\nabla z \cdot \nabla z\partial_t\xi e^\xi dx \\
&= -\int_\vartheta |\partial_t z|^2 e^\xi dx - \int_\vartheta A\nabla z \cdot \nabla\xi\partial_t ze^\xi dx + \int_\vartheta g\partial_t ze^\xi dx + \frac{1}{2}\int_\vartheta A\nabla z \cdot \nabla z\partial_t\xi e^\xi dx \\
&= -\int_\vartheta \left|\partial_t z + A\nabla z \cdot \nabla\xi - \frac{1}{2}g\right|^2 e^\xi dx + \frac{1}{2}\int_\vartheta A\nabla z \cdot \nabla z\partial_t\xi e^\xi dx \\
&\quad + \int_\vartheta (\partial_t z - g) A\nabla z \cdot \nabla\xi e^\xi dx + \int_\vartheta |A\nabla z \cdot \nabla\xi|^2 e^\xi dx + \int_\vartheta \left|\frac{1}{2}g\right|^2 e^\xi dx\ .
\end{aligned}$$

We compute $\int_{\vartheta} (\partial_t z - g)\, A\nabla z \cdot \nabla \xi e^{\xi} dx$ by taking $\varphi = A\nabla z \cdot \nabla \xi e^{\xi}$: One has with standard summation notations and $A = (A_{ij})_{1\leq i,j\leq n}$

$$\begin{array}{l}
\int_{\vartheta} (\partial_t z - g)\, A\nabla z \cdot \nabla \xi e^{\xi} dx + \int_{\vartheta} \left| A\nabla z \cdot \nabla \xi \right|^2 e^{\xi} dx \\
= -\int_{\vartheta} A\nabla z \cdot \nabla \left(A\nabla z \cdot \nabla \xi e^{\xi} \right) dx + \int_{\partial\vartheta} (A\nabla z \cdot \nu)\,(A\nabla z \cdot \nabla \xi)\, e^{\xi} dx + \int_{\vartheta} \left| A\nabla z \cdot \nabla \xi \right|^2 e^{\xi} dx \\
= -\int_{\vartheta} A_{ij}\partial_{x_j} z \partial_{x_i} A_{k\ell} \partial_{x_\ell} z \partial_{x_k} \xi e^{\xi} dx - \int_{\vartheta} A\nabla^2 \xi A\nabla z \cdot \nabla z e^{\xi} dx \\
\quad -\int_{\vartheta} A\nabla^2 z A\nabla z \cdot \nabla \xi e^{\xi} dx + \int_{\partial\vartheta} (A\nabla z \cdot \nu)\,(A\nabla z \cdot \nabla \xi)\, e^{\xi} dx \ .
\end{array}$$

But by one integration by parts

$$\begin{array}{l}
\quad -\int_{\vartheta} A\nabla^2 z A\nabla z \cdot \nabla \xi e^{\xi} dx \\
= -\frac{1}{2}\int_{\partial\vartheta} (A\nabla z \cdot \nabla z)\,(A\nabla \xi \cdot \nu)\, e^{\xi} dx + \frac{1}{2}\int_{\vartheta} \partial_{x_\ell} A_{ij} \partial_{x_j} z A_{k\ell} \partial_{x_i} z \partial_{x_k} \xi e^{\xi} dx \\
\quad +\frac{1}{2}\int_{\vartheta} (A\nabla z \cdot \nabla z)\, \nabla \cdot (A\nabla \xi)\, e^{\xi} dx + \frac{1}{2}\int_{\vartheta} (A\nabla z \cdot \nabla z)\,(A\nabla \xi \cdot \xi)\, e^{\xi} dx \ .
\end{array}$$

The homogeneous Dirichlet boundary condition on z implies $\nabla z = \nu \partial_\nu z$ on $\partial\vartheta$. Therefore, one deduces

$$\begin{array}{ll}
\frac{1}{2}\frac{d}{dt}\int_{\vartheta} A\nabla z \cdot \nabla z e^{\xi} dx & = \frac{1}{2}\int_{\vartheta} \partial_t A\nabla z \cdot \nabla z e^{\xi} dx - \int_{\vartheta} \left| \partial_t z + A\nabla z \cdot \nabla \xi - \frac{1}{2} g \right|^2 e^{\xi} dx \\
& \quad -\int_{\vartheta} A\nabla^2 \xi A\nabla z \cdot \nabla z e^{\xi} dx \\
& \quad +\frac{1}{2}\int_{\vartheta} (A\nabla z \cdot \nabla z)\,(\partial_t \xi + \nabla \cdot (A\nabla \xi) + A\nabla \xi \cdot \nabla \xi)\, e^{\xi} dx \\
& \quad -\int_{\vartheta} A_{ij}\partial_{x_j} z \partial_{x_i} A_{k\ell} \partial_{x_\ell} z \partial_{x_k} \xi e^{\xi} dx + \frac{1}{2}\int_{\vartheta} \partial_{x_\ell} A_{ij} \partial_{x_j} z A_{k\ell} \partial_{x_i} z \partial_{x_k} \xi e^{\xi} dx \\
& \quad +\frac{1}{2}\int_{\partial\vartheta} (A\nabla z \cdot \nabla z)\,(A\nabla \xi \cdot \nu)\, e^{\xi} dx + \int_{\vartheta} \left| \frac{1}{2} g \right|^2 e^{\xi} dx \ .
\end{array}$$

Now, we are able to compute $\frac{d}{dt}\frac{\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx}$: One has

$$\begin{aligned}
&\left(\int_\vartheta |z|^2 e^\xi dx\right)^2 \frac{d}{dt}\frac{\displaystyle\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx}{\displaystyle\int_\vartheta |z|^2 e^\xi dx}\\
&= -2\int_\vartheta A\nabla^2\xi A\nabla z\cdot\nabla z e^\xi dx\int_\vartheta |z|^2 e^\xi dx + \int_{\partial\vartheta}\left(A\nabla z\cdot\nabla z\right)\left(A\nabla\xi\cdot\nu\right)e^\xi dx\int_\vartheta |z|^2 e^\xi dx\\
&\quad -2\int_\vartheta \left|\partial_t z + A\nabla z\cdot\nabla\xi - \frac{1}{2}g\right|^2 e^\xi dx\int_\vartheta |z|^2 e^\xi dx\\
&\quad +2\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx\left(\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx - \int_\vartheta g z e^\xi dx\right)\\
&\quad +\int_\vartheta \partial_t A\nabla z\cdot\nabla z e^\xi dx\int_\vartheta |z|^2 e^\xi dx\\
&\quad +2\left(-\int_\vartheta A_{ij}\partial_{x_j}z\partial_{x_i}A_{k\ell}\partial_{x_\ell}z\partial_{x_k}\xi e^\xi dx + \frac{1}{2}\int_\vartheta \partial_{x_\ell}A_{ij}\partial_{x_j}zA_{k\ell}\partial_{x_i}z\partial_{x_k}\xi e^\xi dx\right)\int_\vartheta |z|^2 e^\xi dx\\
&\quad +\int_\vartheta \left(A\nabla z\cdot\nabla z\right)\left(\partial_t\xi + \nabla\cdot\left(A\nabla\xi\right) + A\nabla\xi\cdot\nabla\xi\right)e^\xi dx\int_\vartheta |z|^2 e^\xi dx\\
&\quad -\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx\left(\int_\vartheta |z|^2\left(\partial_t\xi + \nabla\cdot\left(A\nabla\xi\right) + A\nabla\xi\cdot\nabla\xi\right)e^\xi dx\right)\\
&\quad +2\int_\vartheta\left|\frac{1}{2}g\right|^2 e^\xi dx\int_\vartheta |z|^2 e^\xi dx\ .
\end{aligned}$$

Notice that by Cauchy-Schwarz inequality, the contribution of the fourth and fifth terms of the above becomes

$$\begin{aligned}
&-\int_\vartheta \left|\partial_t z + A\nabla z\cdot\nabla\xi - \frac{1}{2}g\right|^2 e^\xi dx\int_\vartheta |z|^2 e^\xi dx\\
&\quad +\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx\left(\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx - \int_\vartheta g z e^\xi dx\right)\\
&= -\int_\vartheta \left|\partial_t z + A\nabla z\cdot\nabla\xi - \frac{1}{2}g\right|^2 e^\xi dx\int_\vartheta |z|^2 e^\xi dx\\
&\quad +\left(-\int_\vartheta\left(\partial_t z + A\nabla z\cdot\nabla\xi - \frac{1}{2}g\right)z e^\xi dx + \frac{1}{2}\int_\vartheta g z e^\xi dx\right)\\
&\quad \times\left(-\int_\vartheta\left(\partial_t z + A\nabla z\cdot\nabla\xi - \frac{1}{2}g\right)z e^\xi dx - \frac{1}{2}\int_\vartheta g z e^\xi dx\right)\\
&= -\int_\vartheta \left|\partial_t z + A\nabla z\cdot\nabla\xi - \frac{1}{2}g\right|^2 e^\xi dx\int_\vartheta |z|^2 e^\xi dx\\
&\quad +\left(\int_\vartheta\left(\partial_t z + A\nabla z\cdot\nabla\xi - \frac{1}{2}g\right)z e^\xi dx\right)^2 - \left(\frac{1}{2}\int_\vartheta g z e^\xi dx\right)^2\\
&\leq 0\ .
\end{aligned}$$

Therefore, one conclude that

$$
\begin{aligned}
\frac{d}{dt}\frac{\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx} \leq & \frac{-2\int_\vartheta A\nabla^2\xi A\nabla z\cdot\nabla z e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx} + \frac{\int_{\partial\vartheta}(A\nabla z\cdot\nabla z)(A\nabla\xi\cdot\nu)e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx} \\
& + \frac{\int_\vartheta |g|^2 e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx} + \frac{\int_\vartheta \partial_t A\nabla z\cdot\nabla z e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx} \\
& + \frac{-2\int_\vartheta A_{ij}\partial_{x_j}z\partial_{x_i}A_{k\ell}\partial_{x_\ell}z\partial_{x_k}\xi e^\xi dx + \int_\vartheta \partial_{x_\ell}A_{ij}\partial_{x_j}zA_{k\ell}\partial_{x_i}z\partial_{x_k}\xi e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx} \\
& + \frac{\int_\vartheta A\nabla z\cdot\nabla z\left(\partial_t\xi+\nabla\cdot(A\nabla\xi)+A\nabla\xi\cdot\nabla\xi\right)e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx} \\
& - \frac{\int_\vartheta A\nabla z\cdot\nabla z e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx} \times \frac{\int_\vartheta |z|^2\left(\partial_t\xi+\nabla\cdot(A\nabla\xi)+A\nabla\xi\cdot\nabla\xi\right)e^\xi dx}{\int_\vartheta |z|^2 e^\xi dx}\ .
\end{aligned}
$$

4.5 Proof of Lemma 4.7

Let $0<\varepsilon<\rho/2$ and $\phi\in C_0^\infty(B_\rho)$ be such that $0\leq\phi\leq 1$, $\phi=1$ on $\{x; d(x,x_0)\leq\rho-\varepsilon\}$. We multiply the equation $\partial_t u-\nabla\cdot(A\nabla u)=0$ by $\phi^2 u e^{-d(x,x_0)^2/h}$ where $h>0$ and integrate over $\Omega\cap B_\rho$. We get by one integration by parts

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_\rho}|\phi u|^2 e^{-d(x,x_0)^2/h}dx + \int_{\Omega\cap B_\rho} A\nabla u\cdot\nabla\left(\phi^2 u e^{-d(x,x_0)^2/h}\right)dx = 0\ .
$$

But, $A\nabla u\cdot\nabla\left(\phi^2 u e^{-d^2/h}\right) = \left[2\phi u A\nabla\phi\cdot\nabla u + \phi^2 A\nabla u\cdot\nabla u + \phi^2 u\left(-\frac{2d\nabla d}{h}\right)\cdot A\nabla u\right]e^{-d^2/h}$. Therefore, by Cauchy-Schwarz inequality, it follows that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_\rho}|\phi u|^2 e^{-d(x,x_0)^2/h}dx + \int_{\Omega\cap B_\rho}\phi^2 A\nabla u\cdot\nabla u e^{-d(x,x_0)^2/h}dx \\
\leq & \int_{\Omega\cap B_\rho}\phi^2 A\nabla u\cdot\nabla u e^{-d(x,x_0)^2/h}dx + \frac{1}{2}\int_{\Omega\cap B_\rho}4A\nabla\phi\cdot\nabla\phi\,|u|^2 e^{-d(x,x_0)^2/h}dx \\
& + \frac{1}{2}\int_{\Omega\cap B_\rho}\frac{4d^2}{h^2}A\nabla d\cdot\nabla d\,|\phi u|^2 e^{-d(x,x_0)^2/h}dx\ .
\end{aligned}
$$

Thus, with the fact that $A_T(x)\nabla d(x,x_0)\cdot\nabla d(x,x_0)=1$, one get for some constant $C_A>0$

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega\cap B_\rho}|\phi u|^2 e^{-d(x,x_0)^2/h}dx-\frac{\rho^2}{h^2}C_A\int_{\Omega\cap B_\rho}|\phi u|^2 e^{-d(x,x_0)^2/h}dx\\&\leq C_A e^{-\frac{(\rho-\varepsilon)^2}{h}}\int_{\Omega\cap B_\rho}|u(x,t)|^2dx\ .\end{aligned}$$

Then we have,

$$\begin{aligned}\int_{\Omega\cap B_\rho}|\phi u(\cdot,T)|^2 e^{-d(x,x_0)^2/h}dx\ &\leq e^{\frac{\rho^2}{h^2}C_A(T-t)}\int_{\Omega\cap B_\rho}|\phi u(\cdot,t)|^2 e^{-d(x,x_0)^2/h}dx\\&\quad+C_A e^{\frac{\rho^2}{h^2}C_A(T-t)}e^{-\frac{(\rho-\varepsilon)^2}{h}}\int_t^T\int_{\Omega\cap B_\rho}|u|^2dxds\end{aligned}$$

which gives

$$\begin{aligned}\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2dx\ &\leq e^{\frac{\rho^2}{h^2}C_A(T-t)}e^{\frac{(\rho-2\varepsilon)^2}{h}}\int_{\Omega\cap B_\rho}|u(x,t)|^2dx\\&\quad+C_A e^{\frac{\rho^2}{h^2}C_A(T-t)}e^{-\frac{(\rho-\varepsilon)^2}{h}}e^{\frac{(\rho-2\varepsilon)^2}{h}}\int_t^T\int_{\Omega\cap B_\rho}|u|^2dxds\ .\end{aligned}$$

Let $T/2<T-\delta h\leq t\leq T$, it yields

$$\begin{aligned}\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2dx\ &\leq e^{CT}e^{\frac{\rho^2}{h}\delta C_A}e^{\frac{(\rho-2\varepsilon)^2}{h}}\int_{\Omega\cap B_\rho}|u(x,t)|^2dx\\&\quad+C_A e^{\frac{\rho^2}{h}\delta C_A}e^{-\frac{(\rho-\varepsilon)^2}{h}}e^{\frac{(\rho-2\varepsilon)^2}{h}}\int_{T-\delta h}^T\int_{\Omega\cap B_\rho}|u|^2dxds\ .\end{aligned}$$

Choose

$$\delta=\frac{1}{C_A}\frac{\varepsilon(2\rho-3\varepsilon)}{2\rho^2}$$

that is $\delta C_A=\frac{1}{2}\frac{(\rho-\varepsilon)^2-(\rho-2\varepsilon)^2}{\rho^2}\in(0,1/8]$ in order that $\rho^2\delta C_A-(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2<0$. Therefore, we get

$$\begin{aligned}\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2dx\ &\leq e^{\frac{(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_{\Omega\cap B_\rho}|u(x,t)|^2dx\\&\quad+C_A e^{\frac{-(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_{T-\delta h}^T\int_{\Omega\cap B_\rho}|u|^2dxdt\\&\leq e^{\frac{(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_{\Omega\cap B_\rho}|u(x,t)|^2dx\\&\quad+C_A e^{\frac{-(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_\Omega|u(x,0)|^2dx\end{aligned}$$

where in the last line we used $\delta h<$max$(1,T/2)$. Now, choose h such that both $\delta h<$max$(1,T/2)$ and

$$(1+C_A)e^{\frac{-(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_\Omega|u(x,0)|^2dx\leq\frac{1}{e}\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2dx\ .$$

With such choice, one has

$$\left(1-\frac{1}{e}\right)\int_{\Omega\cap B_{\rho-2\varepsilon}}\left|u\left(x,T\right)\right|^2dx\leq e^{\frac{(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_{\Omega\cap B_\rho}\left|u\left(x,t\right)\right|^2dx$$

and moreover,

$$\int_\Omega\left|u\left(x,0\right)\right|^2dx\leq e^{\frac{(\rho-\varepsilon)^2}{h}}\int_{\Omega\cap B_\rho}\left|u\left(x,t\right)\right|^2dx$$

for any $T/2<T-\delta h\leq t\leq T$. Such h exists by choosing

$$h=\frac{\varepsilon\left(2\rho-3\varepsilon\right)/2}{\ln\left(K\dfrac{(1+C_A)\displaystyle\int_\Omega\left|u\left(x,0\right)\right|^2dx}{\dfrac{1}{e}\displaystyle\int_{\Omega\cap B_{\rho-2\varepsilon}}\left|u\left(x,T\right)\right|^2dx}\right)}\text{ with }K=e^{\varepsilon\frac{(2\rho-3\varepsilon)}{2}\left(\frac{2}{T}+1\right)\delta}\ .$$

Clearly, $\delta h<T/2$ and $\delta h\leq 1$. We conclude that for any $T/2\leq T-\theta\leq t\leq T$

$$\frac{\displaystyle\int_\Omega\left|u\left(x,0\right)\right|^2dx}{\displaystyle\int_{\Omega\cap B_\rho}\left|u\left(x,t\right)\right|^2dx}\leq e^{\frac{1}{C_A}\frac{\varepsilon(2\rho-3\varepsilon)(\rho-\varepsilon)^2}{2\rho^2}\frac{1}{\theta}}$$

with

$$\frac{1}{\theta}=C_A\frac{4\rho^2}{\varepsilon^2\left(2\rho-3\varepsilon\right)^2}\ln\left(e\left(1+C_A\right)e^{\left(\frac{2}{T}+1\right)\frac{1}{C_A}\frac{\varepsilon^2(2\rho-3\varepsilon)^2}{4\rho^2}}\frac{\displaystyle\int_\Omega\left|u\left(x,0\right)\right|^2dx}{\displaystyle\int_{\Omega\cap B_{\rho-2\varepsilon}}\left|u\left(x,T\right)\right|^2dx}\right)\ .$$

This completes the proof.

Remark .- When A is time-independent, then $C_A=4\max\Big(1,\left\|A\nabla\phi\cdot\nabla\phi\right\|_{L^\infty(\Omega)}\Big)$.

Appendix

This appendix is devoted to the proof of Proposition 3.2 and of inequality (2.2).

Trace estimate for f (proof of Proposition 3.2)

Denote $\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)_+ = \left\{(x,v)\in\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}; v\cdot\vec{n}_x\geq 0\right\}$. First, multiplying both sides of the first line of (1.1) by $\eta f\left|f\right|^{\eta-2}$ and integrating over $\Omega\times\mathbb{S}^{\mathrm{d}-1}\times(0,T)$, one has the following a priori estimate for any $\eta\geq 2$

$$\int_0^T\int_{\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)_+} v\cdot\vec{n}_x\left|f\right|^{\eta}dxdvdt\leq\epsilon\frac{2}{\eta}\int_{\Omega\times\mathbb{S}^{\mathrm{d}-1}}\left|f_0\right|^{\eta}dxdv\ .$$

Secondly, one uses Hölder inequality to get

$$\begin{aligned}&\int_0^T\int_{\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}}\left|f\right|^2dxdvdt\\ \leq&\left(\int_0^T\int_{\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)_+}\frac{dxdvdt}{\left(v\cdot\vec{n}_x\right)^{\frac{1}{p-1}}}\right)^{\frac{p-1}{p}}\left(\int_0^T\int_{\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)_+}v\cdot\vec{n}_x\left|f\right|^{2p}dxdvdt\right)^{\frac{1}{p}}\ .\end{aligned}$$

But

$$\int_{\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\right)_+}\frac{dxdv}{\left(v\cdot\vec{n}_x\right)^{\frac{1}{p-1}}}\leq C\frac{p-1}{p-2}\text{ for any }p>2\ .$$

Hence, as soon as $p>2$, one get the desired estimate

$$\|f\|_{L^2\left(\partial\Omega\times\mathbb{S}^{\mathrm{d}-1}\times(0,T)\right)}\leq CT^{\frac{p-1}{2p}}\epsilon^{\frac{1}{2p}}C_p\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{\mathrm{d}-1})}$$

where $C_p=\left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}}\left(\frac{1}{p}\right)^{\frac{1}{2p}}$ and $C>0$ only depends on (Ω,d).

Backward estimate for diffusion equations (proof of (2.2))

Classical energy identities for our parabolic equation are:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\left|u\right|^2dx+\int_\Omega\frac{1}{\mathrm{da}}\left|\nabla u\right|^2dx=0\ ,$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{1}{\mathrm{da}}\left|\nabla\varphi\right|^2dx+\int_\Omega\left|u\right|^2dx=0\ ,$$

where $\varphi\left(\cdot,t\right)\in H_0^1\left(\Omega\right)$ solves $-\nabla\cdot\left(\frac{1}{\mathrm{da}}\nabla\varphi\left(\cdot,t\right)\right)=u\left(\cdot,t\right)$ in Ω. Now, one can easily check with $y\left(t\right)=\int_\Omega\frac{1}{\mathrm{da}\left(x\right)}\left|\nabla\varphi\left(x,t\right)\right|^2dx$ and $N\left(t\right)=\dfrac{\int_\Omega\left|u\left(x,t\right)\right|^2dx}{\int_\Omega\frac{1}{\mathrm{da}\left(x\right)}\left|\nabla\varphi\left(x,t\right)\right|^2dx}$ that

$$\left\{\begin{array}{l}\frac{1}{2}y'\left(t\right)+N\left(t\right)y\left(t\right)=0\\ N'\left(t\right)\leq 0\ .\end{array}\right.$$

By solving such differential inequalities, one obtain

$$\int_{\Omega} \frac{1}{\mathrm{d}a(x)} |\nabla\varphi(x,0)|^2 \, dx \leq e^{2TN(0)} \int_{\Omega} \frac{1}{\mathrm{d}a(x)} |\nabla\varphi(x,T)|^2 \, dx$$

which implies

$$\|u(\cdot,T)\|^2_{H^{-1}(\Omega)} \leq \frac{c_{max}}{c_{min}} e^{2T \frac{\|u(\cdot,0)\|^2_{L^2(\Omega)}}{\mathrm{d}c_{min}\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}} \|u(\cdot,T)\|^2_{H^{-1}(\Omega)} \ .$$

One conclude that

$$\|u(\cdot,0)\|_{L^2(\Omega)} \leq ce^{cT \frac{\|u(\cdot,0)\|^2_{L^2(\Omega)}}{\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}} \|u(\cdot,T)\|_{L^2(\Omega)} \ .$$

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